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A model for electroelastic plates under biasing fields with applications in buckling analysis

Y.T. Hu [a,b,*], J.S. Yang [c], Q. Jiang [d]

[a] Department of Mechanics, Huazhong University of Science and Technology, Wuhan 430074, China
[b] State Key Laboratory of Structural Analysis of Industrial Equipment, Dalian University of Technology, Dalian 116023, China
[c] Department of Engineering Mechanics, University of Nebraska, Lincoln, NE 68588, USA
[d] Department of Mechanical Engineering, University of California, Riverside, CA 92521, USA

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Abstract

This paper presents a theoretical model for coupled extension and flexure with shear deformations of an electroelastic plate under biasing fields. The governing equations of this model, defined in the middle plane of the plates, are derived from the full three-dimensional theory of electroelasticity for small fields superposed upon finite biasing fields, under the assumption that the stress component normal to the plate vanishes identically. As examples to illustrate the applications of this model, the authors include their analysis of buckling of three plates, one single-layered plate and two double-layered plates (i.e., bimorphs) of distinct poling configurations. This analysis indicates that the electromechanical coupling strengthens the plates against buckling.

Keywords: Buckling; Flexure; Shear deformations; Electroelastic

1. Introduction

Piezoelectric effect often refers to the linear coupling between mechanical deformations and electric fields. In contrast, materials exhibiting nonlinear electromechanical coupling are called electroelastic materials. An example of such materials is the electrostrictive materials which are characterized by their quadratic dependence of mechanical fields on electric fields. Many electromechanical devices have components made of electroelastic materials, and some of these components can be modeled as plates because of their plate-like geometry. The investigations on the governing equations of piezoelectric plates were initiated by Mindlin, and the early contributions by Mindlin and his students can be found in the books by Tiersten (1969) and Deresiewicz et al. (1989), as well as a review article by Wang and Yang (2000). Recently, the study of smart structures further enriches the theories of piezoelectric plates, and many references on this topic can be found in Rao and Sunar (1994), Tani et al. (1998), and Sunar and Rao (1999).

* Corresponding author. Address: Department of Mechanics, Huazhong University of Science and Technology, Wuhan 430074, China. Tel.: +86-27-8754-4334; fax: +86-27-8754-5524.
E-mail address: hudeng@263.net (Y.T. Hu).

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The performance of electromechanical devices can be affected by many environmental effects that act as initial or biasing fields in these devices. For example, for a piezoelectric resonator designed to operate at a particular resonant frequency, a temperature change or mounting stresses can cause initial deformations and stresses in the resonator and induce frequency shift (Tiersten and Sinha, 1979). The performance of underwater transducers is affected by initial fields due to hydrostatic pressure (Wilson, 1985). A piezoelectric device mounted on a moving object is affected by acceleration induced stresses or strains (Zhou and Tiersten, 1991). While biasing fields can cause many undesirable effects, they are the foundation of the working principles of many piezoelectric sensors (Ballantine et al., 1997). For example, the frequency shift of a piezoelectric resonator caused by a temperature change can be used to design thermometers. The response of an electroelastic body under biasing fields can also be used to measure nonlinear material constants (Cho and Yamanouchi, 1987). Either for the purpose of avoiding the biasing fields or making use of them, knowledge of the behaviors of electroelastic bodies under biasing fields is crucial in the design of many electromechanical devices. The effect of biasing fields in an electroelastic body can be described by the theory of small fields superposed on finite biasing fields. Such a theory was given by Baumhauer and Tiersten (1973), which can also be found in a more recent paper (Tiersten, 1995). The development of the theory for small fields on finite biasing fields relies on the fully nonlinear theory of electroelasticity (Tiersten, 1971).

In the special case of an elastic body under biasing fields, many problems have been studied, which can be found in Iesan's book (1989) with applications in, e.g., stability. In particular, two-dimensional equations for anisotropic elastic plates under various biasing fields were derived by a few investigators (Lee et al., 1975; Lee and Yong, 1986; Wang et al., 1998; Wang, 1999) using power series or trigonometric approximations along the plate thickness. In this paper, we develop a theoretical model for coupled extension and flexure with shear deformations of an electroelastic plate under biasing fields. The governing equations of this model are derived from the full three-dimensional theory of electroelasticity for small fields superposed upon finite biasing fields, under the assumption that the stress component normal to the plate vanishes identically. To illustrate the applications of this model, we include the analysis of buckling of three plates, one single-layered plate and two double-layered plates (i.e., bimorphs) of distinct poling configurations.

2. Equations for small fields superposed on finite biasing fields

The theory for small fields superposed on finite biasing fields in an electroelastic body (Baumhauer and Tiersten, 1973; Tiersten, 1995) is summarized in this section. Consider the following three configurations of an electroelastic body as shown in Fig. 1. Cartesian tensor notation, the summation convention for re-

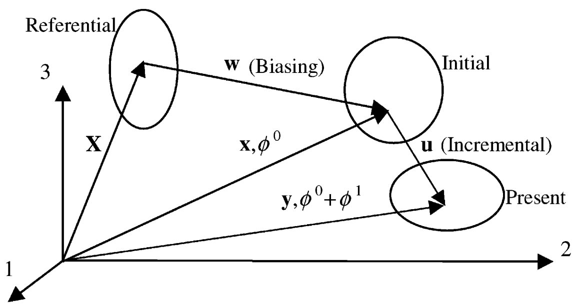

Fig. 1. The referential, initial, and present configurations of an electroelastic body.

peated tensor indices and the convention that a comma followed by an index denotes partial differentiation with respect to the coordinate associated with the index will be used. A superimposed dot represents material time derivative.

2.1. The referential configuration

At time $t = 0$, the body is undeformed and free of all fields. A generic point at this state is denoted by $\mathbf{X}$ having rectangular coordinates X_K. The mass density in the referential configuration is denoted by ρ_0.

2.2. The initial configuration

In this state the body is deformed finitely and statically, and carries finite, static electric fields. The position of the material point associated with $\mathbf{X}$ is given by $x_\alpha = x_\alpha(\mathbf{X})$, with $\mathbf{w} = \mathbf{x} - \mathbf{X}$ as the initial displacement vector. The electric potential of this state is denoted by ϕ^0. The initial deformations and fields are also called the biasing fields. They satisfy the following static equations of nonlinear electroelasticity

$$\widehat{T}^0_{K\alpha,K} = 0, \quad \widehat{D}^0_{K,K} = 0, \tag{1}$$

where the body force and charge are not included. The total Piola–Kirchhoff stress tensor $\widehat{T}^0_{K\alpha}$ and the referential electric displacement vector $\widehat{D}^0_K$ have the expressions

$$\begin{aligned} \widehat{T}^0_{K\alpha} &= x_{\alpha,L}T^0_{KL} + J_0 X_{K,\beta}\varepsilon_0\left(E^0_\beta E^0_\alpha - E^0_\gamma E^0_\gamma \delta_{\beta\alpha}/2\right), \\ \widehat{D}^0_K &= \varepsilon_0 J_0 X_{K,\alpha}E^0_\alpha + \widehat{P}^0_K. \end{aligned} \tag{2}$$

In (2), $J_0 = \det(x_{\alpha,K})$, $E^0_\alpha = -\phi^0_{,\alpha}$, $\delta_{\alpha\beta}$ is the Kronecker delta, ε_0 the dielectric permittivity of free space, and

$$\widehat{T}^0_{KL} = \left.\frac{\partial \Sigma}{\partial S_{KL}}\right|_{\mathbf{E}^0,\mathbf{W}^0}, \quad \widehat{P}^0_K = -\left.\frac{\partial \Sigma}{\partial W_K}\right|_{\mathbf{E}^0,\mathbf{W}^0}. \tag{3}$$

In (3), the finite strains and the referential electric field of the initial configuration are defined by

$$S^0_{KL} = (x_{\alpha,K}x_{\alpha,L} - \delta_{KL})/2, \quad W^0_K = -\phi^0_{,K} \tag{4}$$

and $\Sigma = \Sigma(S_{KL}, W_K)$ is an energy density function of the strain tensor $\mathbf{S}$ and the referential electric field $\mathbf{W}$. When the fields are moderately large, the weak nonlinear behavior of the material can be described by the lower-order terms of

$$\begin{aligned} \Sigma(S_{KL}, W_K) = {} & \tfrac{1}{2}c_{ABCD}S_{AB}S_{CD} - e_{ABC}W_A S_{BC} - \tfrac{1}{2}\varepsilon_0\chi_{AB}W_A W_B + \tfrac{1}{6}c_{ABCDEF}S_{AB}S_{CD}S_{EF} + \tfrac{1}{2}k_{ABCDE}W_A S_{BC}S_{DE} \\ & - \tfrac{1}{2}b_{ABCD}W_A W_B S_{CD} - \tfrac{1}{6}\chi_{ABC}W_A W_B W_C + \text{higher-order terms}, \end{aligned} \tag{5}$$

where the second-order material constants c_{ABCD} and e_{ABC} represent the elastic and piezoelectric constants, and χ_{AB} the dielectric susceptibility. They are responsible for linear material behaviors. c_{ABCDEF}, k_{ABCDE} and χ_{ABC} are the third-order elastic, piezoelectric, and dielectric constants. b_{ABCD} are the electrostrictive constants. The third- and higher-order material constants are related to nonlinear behaviors of the material. The material constants in (5) are called the fundamental material constants, in comparison to the effective material constants to be introduced below.

2.3. The present configuration

To the deformed body at the initial configuration, time-dependent small deformations and electric fields are applied. The final position of the material point associated with $\mathbf{X}$ is given by $y_i = y_i(\mathbf{X}, t)$. The small, incremental displacement vector is denoted by $\mathbf{u}$, and the incremental electric potential by ϕ^1. The equations of motion and electrostatics for the incremental fields are

$$\widehat{T}^1_{K\alpha,K} = \rho_0\ddot{u}_\alpha, \quad \widehat{D}^1_{K,K} = 0, \tag{6}$$

in which the incremental stress tensor and incremental electric displacement vector are given by the following constitutive relations

$$\begin{aligned} \widehat{T}^1_{K\alpha} &= G_{K\alpha L\gamma}u_{\gamma,L} + R_{LK\alpha}\phi^1_{,L}, \\ \widehat{D}^1_K &= R_{KL\gamma}u_{\gamma,L} - L_{KL}\phi^1_{,L}. \end{aligned} \tag{7}$$

In (7), $G_{K\alpha L\gamma}$, $R_{LK\alpha}$, and L_{KL} are the effective elastic, piezoelectric, and dielectric constants with the following expressions

$$\begin{aligned} G_{K\alpha L\gamma} &= x_{\alpha,M}\frac{\partial^2\Sigma}{\partial S_{KM}\partial S_{LN}}\bigg|_{\mathbf{E}^0,\mathbf{W}^0} x_{\gamma,N} + T^0_{KL}\delta_{\alpha\gamma} + g_{K\alpha L\gamma} = G_{L\gamma K\alpha}, \\ R_{KL\gamma} &= -\frac{\partial^2\Sigma}{\partial W_K\partial S_{ML}}\bigg|_{\mathbf{E}^0,\mathbf{W}^0} x_{\gamma,M} + r_{KL\gamma}, \\ L_{KL} &= -\frac{\partial^2\Sigma}{\partial W_K\partial W_L}\bigg|_{\mathbf{E}^0,\mathbf{W}^0} + l_{KL} = L_{LK}, \end{aligned} \tag{8}$$

where

$$\begin{aligned} g_{K\alpha L\gamma} = \varepsilon_0 J_0\Big[&E^0_\alpha E^0_\beta\big(X_{K,\beta}X_{L,\gamma} - X_{K,\gamma}X_{L,\beta}\big) - E^0_\alpha E^0_\gamma X_{K,\beta}X_{L,\beta} + E^0_\beta E^0_\gamma\big(X_{K,\alpha}X_{L,\beta} - X_{K,\beta}X_{L,\alpha}\big) \\ &+ \tfrac{1}{2}E^0_\beta E^0_\beta\big(X_{K,\gamma}X_{L,\alpha} - X_{K,\alpha}X_{L,\gamma}\big)\Big], \end{aligned} \tag{9a}$$

$$r_{KL\gamma} = \varepsilon_0 J_0\Big(E^0_\alpha X_{K,\alpha}X_{L,\gamma} - E^0_\alpha X_{K,\gamma}X_{L,\alpha} - E^0_\gamma X_{K,\alpha}X_{L,\alpha}\Big), \tag{9b}$$

$$l_{KL} = \varepsilon_0 J_0 X_{K,\alpha}X_{L,\alpha}. \tag{9c}$$

We note that in (7) the incremental stress tensor and electric displacement vector depend linearly on the incremental displacement gradient and potential gradient. The important thing to observe is that the effective material constants are functions of the linear and nonlinear material constants c_{ABCD}, e_{ABC}, χ_{AB}, c_{ABCDEF}, k_{ABCDE}, χ_{ABC}, b_{ABCD}, etc., and the biasing fields $\mathbf{w}$ and ϕ^0. Since the incremental stress tensor and displacement gradient in general are not symmetric and that the effective constants depend on the biasing fields, the effective material constants in (9a)–(9c) usually have lower symmetry than the fundamental linear elastic, piezoelectric, and dielectric constants. This is called the induced anisotropy. There can be as many as 45 independent components for $G_{K\alpha L\gamma}$, 27 independent components for $R_{KL\gamma}$, and six independent components for L_{KL}. The above shows that the biasing fields effectively change the material properties and hence the material response to excitations. We note that the expression in Tiersten (1995) corresponding to (9a) has minor algebraic errors (Yang and Tiersten, 1999), and (9a) is the corrected version. The following variational formulation corresponding to (6) is convenient for developing plate equations

$$\int_V \Big[\Big(\widehat{T}^1_{K\alpha,K} - \rho_0\ddot{u}_\alpha\Big)\delta u_\alpha + \widehat{D}^1_{K,K}\delta\phi^1\Big]\mathrm{d}V = 0. \tag{10}$$

3. The derivation of plate equations

Consider an electroelastic plate in the reference configuration with the X_3 axis along the plate normal, as shown in Fig. 2. Since the plate is assumed to be thin, we make the usual assumption of vanishing normal stress (Mindlin, 1955)

$$\widehat{T}^1_{33} = G_{33L\gamma}u_{\gamma,L} + R_{L33}\phi^1_{,L} = 0. \tag{11}$$

From (11) we can solve for $u_{3,3}$ with the result

$$u_{3,3} = \frac{-1}{G_{3333}}\left[G_{33L\gamma}u_{\gamma,L} - G_{3333}u_{3,3} + R_{L33}\phi^1_{,L}\right]. \tag{12}$$

We note that $u_{3,3}$ has been eliminated from the right hand side of (12) because when $L = \gamma = 3$ the two terms containing $u_{3,3}$ will cancel with each other. Substituting (12) back into (7), we obtain

$$\begin{aligned}\widehat{T}^1_{K\alpha} &= \overline{G}_{K\alpha L\gamma}u_{\gamma,L} + \overline{R}_{LK\alpha}\phi^1_{,L},\\ \widehat{D}^1_K &= \overline{R}_{KL\gamma}u_{\gamma,L} - \overline{L}_{KL}\phi^1_{,L},\end{aligned} \tag{13}$$

where

$$\begin{aligned}\overline{G}_{K\alpha L\gamma} &= G_{K\alpha L\gamma} - G_{K\alpha 33}G_{33L\gamma}/G_{3333},\\ \overline{R}_{KL\gamma} &= R_{KL\gamma} - R_{K33}G_{33L\gamma}/G_{3333},\\ \overline{L}_{KL} &= L_{KL} + R_{K33}R_{L33}/G_{3333}.\end{aligned} \tag{14}$$

We note that in (13) $\widehat{T}^1_{33} = 0$ and its right hand side does not contain $u_{3,3}$.

For a first-order theory we make the following expansions of the incremental displacement and electric potential

$$\begin{aligned}u_1 &\cong u_1^{(0)}(X_1, X_2, t) + X_3u_1^{(1)}(X_1, X_2, t),\\ u_2 &\cong u_2^{(0)}(X_1, X_2, t) + X_3u_2^{(1)}(X_1, X_2, t),\\ u_3 &\cong u_3^{(0)}(X_1, X_2, t),\\ \phi^1 &\cong \phi^{(0)}(X_1, X_2, t) + X_3\phi^{(1)}(X_1, X_2, t),\end{aligned} \tag{15}$$

where $u_1^{(0)}$ and $u_2^{(0)}$ are the plate extensional displacements, $u_3^{(0)}$ the flexural displacement, and $u_1^{(1)}$ and $u_2^{(1)}$ the shear displacements. From these plate displacements the thickness expansion or contraction accompanying the extension and flexure of the plate due to Poisson's effect can be found from (12) if wanted. Substituting (15) into (10), with integration by parts, for independent variations of $\delta u_1^{(0)}$, $\delta u_2^{(0)}$, $\delta u_3^{(0)}$, $\delta u_1^{(1)}$, $\delta u_2^{(1)}$, $\delta\phi^{(0)}$ and $\delta\phi^{(1)}$, we obtain the following two-dimensional equations of motion and electrostatics

$$\widehat{T}^{(0)}_{A\alpha,A} + F^{(0)}_\alpha = 2\rho_0 h\ddot{u}^{(0)}_\alpha, \quad \alpha = 1,2,3, \tag{16a}$$

$$\widehat{T}^{(1)}_{A\alpha,A} - \widehat{T}^{(0)}_{3\alpha} + F^{(1)}_\alpha = \frac{2\rho_0 h^3}{3}\ddot{u}^{(1)}_\alpha, \quad \alpha = 1,2, \tag{16b}$$

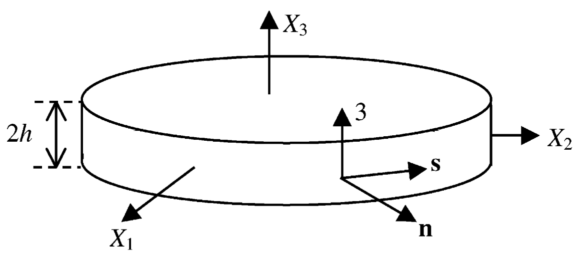

Fig. 2. The reference configuration of an electroelastic plate and the coordinate system.

$$\widehat{D}^{(0)}_{A,A} + \widehat{D}^{(0)} = 0, \tag{16c}$$

$$\widehat{D}^{(1)}_{A,A} - \widehat{D}^{(0)}_{3} + \widehat{D}^{(1)} = 0, \tag{16d}$$

where we have introduced the convention that the index A assumes 1 and 2 but not 3. Eq. (16a) for $\alpha = 1, 2$ are the equations for extension, and for $\alpha = 3$ the equation for flexure. Eq. (16b) are for shears in the X_1 and X_2 directions. In (16a)–(16d) the plate resultants and surface loads of various orders are defined by

$$\left\{\widehat{T}^{(n)}_{K\alpha}, \widehat{D}^{(n)}_{K}\right\} = \int_{-h}^{h} X_3^n \left\{\widehat{T}^{1}_{K\alpha}, \widehat{D}^{1}_{K}\right\} \mathrm{d}X_3, \tag{17a}$$

$$F^{(n)}_{\alpha} = \left[X_3^n \widehat{T}^{1}_{3\alpha}\right]^{h}_{-h}, \qquad \widehat{D}^{(n)} = [X_3^n \widehat{D}^{1}_{3}]^{h}_{-h}, \quad n = 0, 1, \tag{17b}$$

where $\widehat{T}^{(n)}_{K\alpha}$ represent plate extensional and shearing forces, and bending and twisting moments. From (15) we can also write

$$\begin{aligned} u_{\gamma,L} &\cong U^{(0)}_{\gamma L}(X_1, X_2, t) + X_3 U^{(1)}_{\gamma L}(X_1, X_2, t), \\ \phi^{1}_{,L} &\cong -W^{(0)}_{L}(X_1, X_2, t) - X_3 W^{(1)}_{L}(X_1, X_2, t), \end{aligned} \tag{18}$$

where

$$\begin{aligned} &U^{(0)}_{11} = u^{(0)}_{1,1}, \quad U^{(0)}_{12} = u^{(0)}_{1,2}, \quad U^{(0)}_{13} = u^{(1)}_{1}, \\ &U^{(0)}_{21} = u^{(0)}_{2,1}, \quad U^{(0)}_{22} = u^{(0)}_{2,2}, \quad U^{(0)}_{23} = u^{(1)}_{2}, \\ &U^{(0)}_{31} = u^{(0)}_{3,1}, \quad U^{(0)}_{32} = u^{(0)}_{3,2}, \quad U^{(0)}_{33} = 0, \\ &U^{(1)}_{11} = u^{(1)}_{1,1}, \quad U^{(1)}_{12} = u^{(1)}_{1,2}, \quad U^{(1)}_{13} = 0, \\ &U^{(1)}_{21} = u^{(1)}_{2,1}, \quad U^{(1)}_{22} = u^{(1)}_{2,2}, \quad U^{(1)}_{23} = 0, \\ &U^{(1)}_{31} = 0, \quad U^{(1)}_{32} = 0, \quad U^{(1)}_{33} = 0, \\ &W^{(0)}_{1} = -\phi^{(0)}_{,1}, \quad W^{(0)}_{2} = -\phi^{(0)}_{,2}, \quad W^{(0)}_{3} = -\phi^{(1)}, \\ &W^{(1)}_{1} = -\phi^{(1)}_{,1}, \quad W^{(1)}_{2} = -\phi^{(1)}_{,2}, \quad W^{(1)}_{3} = 0. \end{aligned} \tag{19}$$

The displacement gradients of various orders in (19) represent plate strains and bending curvatures, etc. From (17a), with the substitution of (13) and (18), we obtain the plate constitutive relations as

$$\begin{aligned} \widehat{T}^{(0)}_{K\alpha} &= G^{(0)}_{K\alpha L\gamma} U^{(0)}_{\gamma L} + G^{(1)}_{K\alpha L\gamma} U^{(1)}_{\gamma L} - R^{(0)}_{LK\alpha} W^{(0)}_{L} - R^{(1)}_{LK\alpha} W^{(1)}_{L}, \\ \widehat{T}^{(1)}_{K\alpha} &= G^{(1)}_{K\alpha L\gamma} U^{(0)}_{\gamma L} + G^{(2)}_{K\alpha L\gamma} U^{(1)}_{\gamma L} - R^{(1)}_{LK\alpha} W^{(0)}_{L} - R^{(2)}_{LK\alpha} W^{(1)}_{L}, \\ \widehat{D}^{(0)}_{K} &= R^{(0)}_{KL\gamma} U^{(0)}_{\gamma L} + R^{(1)}_{KL\gamma} U^{(1)}_{\gamma L} + L^{(0)}_{KL} W^{(0)}_{L} + L^{(1)}_{KL} W^{(1)}_{L}, \\ \widehat{D}^{(1)}_{K} &= R^{(1)}_{KL\gamma} U^{(0)}_{\gamma L} + R^{(2)}_{KL\gamma} U^{(1)}_{\gamma L} + L^{(1)}_{KL} W^{(0)}_{L} + L^{(2)}_{KL} W^{(1)}_{L}, \end{aligned} \tag{20}$$

where

$$\left\{G^{(n)}_{K\alpha L\gamma}, R^{(n)}_{Kl\gamma}, L^{(n)}_{KL}\right\} = \int_{-h}^{h} X_3^n \{\overline{G}_{K\alpha L\gamma}, \overline{R}_{KL\gamma}, \overline{L}_{KL}\} \mathrm{d}X_3, \quad n = 0, 1, 2. \tag{21}$$

Physically, $G^{(n)}_{K\alpha L\gamma}$ represent the plate flexural and extensional stiffness, etc. We note from (20) that extension and bending may be coupled due to nonuniform biasing fields. We also note from (21) that in a plate theory, only the moments of various orders of the biasing fields matter, not the exact three-dimensional distributions of the biasing fields. In summary, we have obtained the two-dimensional equations of motion and electrostatics (16a)–(16d), the constitutive relations (20), and the displacement gradients and electric fields (19). With successive substitutions, (16a)–(16d) can be written as seven equations for the seven unknowns of $u_1^{(0)}$, $u_2^{(0)}$, $u_3^{(0)}$, $u_1^{(1)}$, $u_2^{(1)}$, $\phi^{(0)}$ and $\phi^{(1)}$. To these equations the proper forms of the boundary conditions can be determined from the variational formulation (10). At the boundary of a plate with unit exterior normal $\mathbf{n}$ and unit tangent $\mathbf{s}$, we need to prescribe

$$\begin{aligned}&\widehat{T}_{nn}^{(0)} \text{ or } u_n^{(0)}, \quad \widehat{T}_{ns}^{(0)} \text{ or } u_s^{(0)}, \quad \widehat{T}_{n3}^{(0)} \text{ or } u_3^{(0)},\\ &\widehat{T}_{nn}^{(1)} \text{ or } u_n^{(1)}, \quad \widehat{T}_{ns}^{(1)} \text{ or } u_s^{(1)},\\ &\widehat{D}_n^{(0)} \text{ or } \phi^{(0)}, \quad \widehat{D}_n^{(1)} \text{ or } \phi^{(1)}.\end{aligned} \tag{22}$$

We note that in the first-order plate theories by Mindlin (1955) using power series approximation for the variation of fields along the plate thickness, shear correction factors are often needed. Those correction factors play a role in shear dominated cases, e.g., vibrations with frequencies close to the first thickness-shear frequency. In the present paper our main interest is on the extension and flexure with small shear deformations. Therefore we ignore the shear correction factors which, if needed, can be included in the manner of Mindlin (1955). The simplest way is to use a modified mass density (Mindlin, 1955).

4. Buckling of piezoelectric plates

Potential applications of the equations derived in the previous section are many. The simplest may be the buckling of electroelastic plates. For the classical description of the buckling phenomenon, the electroelastic counterpart of the initial stress theory in elasticity is sufficient. Such a theory can be obtained by setting $\mathbf{x} = \mathbf{X}$ in the equations for small fields superposed on finite biasing fields (Iesan, 1989). Furthermore, for buckling analysis, a quadratic expression of Σ with the first three terms of the right hand side of (5) and the corresponding linear constitutive relations will be enough, and the biasing fields can be treated as small fields too. With these simplifications, we use the equations in the previous sections to perform buckling analysis of a few piezoelectric plates and bimorphs made from polarized ceramics. Three cases corresponding to Fig. 3 will be considered. We limit our discussion to plane strain analysis with $u_2 = 0$ and $\partial/\partial X_2 = 0$. For all three cases the two major surfaces of the plate at $X_3 = \pm h$ are traction free and are unelectroded with vanishing normal electric displacement. The plates are mechanically simply supported at their end faces at $X_1 = 0$ and $X_1 = l$. The electric end conditions will be specified later in each specific case. The two ends of the plates are loaded by an axial force $p = 2h\widehat{T}_{11}^0$ per unit length in the X_2 direction, which is responsible for the biasing deformation.

4.1. Case (a)

Consider a ceramic plate poled in the X_1 direction as shown in Fig. 3(a). From Auld (1973), the material matrices are

$$\begin{pmatrix} c_{33} & c_{13} & c_{13} & 0 & 0 & 0\\ c_{31} & c_{11} & c_{12} & 0 & 0 & 0\\ c_{31} & c_{21} & c_{11} & 0 & 0 & 0\\ 0 & 0 & 0 & c_{66} & 0 & 0\\ 0 & 0 & 0 & 0 & c_{44} & 0\\ 0 & 0 & 0 & 0 & 0 & c_{44}\end{pmatrix}, \quad \begin{pmatrix} e_{33} & 0 & 0\\ e_{31} & 0 & 0\\ e_{31} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & e_{15}\\ 0 & e_{15} & 0\end{pmatrix}, \quad \begin{pmatrix} \varepsilon_{33} & 0 & 0\\ 0 & \varepsilon_{11} & 0\\ 0 & 0 & \varepsilon_{11}\end{pmatrix}, \tag{23}$$

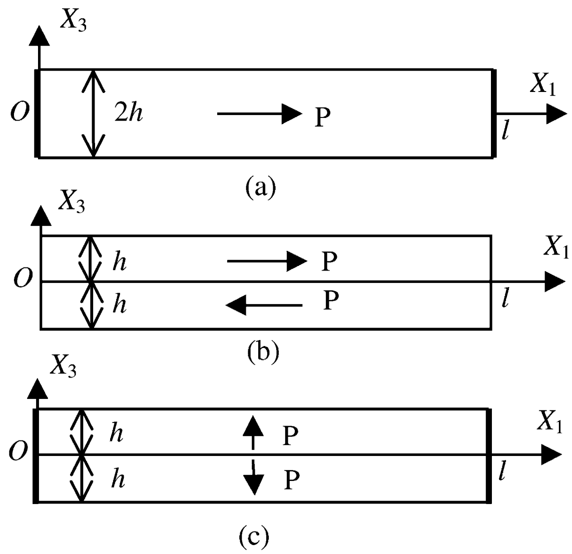

Fig. 3. Simply supported ceramic plates of length l and thickness $2h$.

which are obtained from the material matrices for ceramics poled in the X_3 direction by properly reordering rows and columns. The plate is electroded at its two end faces at $X_1 = 0$ and $X_1 = l$. The electrodes are shown by thick lines in the figure. When the initial load p is being applied, the end electrodes are shorted to eliminate the initial electric field E_1 which otherwise would exist. Once p is already loaded, there exist initial charges on the end electrodes. The electrodes are then opened during the incremental flexural deformation and there are no incremental charges on these electrodes. Therefore for the incremental fields the electric displacement vanishes at both ends. Then the governing equations for the incremental fields take the form

$$G^{(0)}_{1313}u^{(0)}_{3,11} + G^{(0)}_{1331}u^{(1)}_{1,1} + R^{(0)}_{313}\phi^{(1)}_{,1} = 0, \tag{24a}$$

$$G^{(2)}_{1111}u^{(1)}_{1,11} - G^{(0)}_{3113}u^{(0)}_{3,1} - G^{(0)}_{3131}u^{(1)}_{1} + R^{(2)}_{111}\phi^{(1)}_{,11} - R^{(0)}_{331}\phi^{(1)} = 0, \tag{24b}$$

$$L^{(0)}_{11}\phi^{(0)}_{,11} = 0, \tag{24c}$$

$$R^{(2)}_{111}u^{(1)}_{1,11} - R^{(0)}_{313}u^{(0)}_{3,1} - R^{(0)}_{331}u^{(1)}_{1} - L^{(2)}_{11}\phi^{(1)}_{,11} + L^{(0)}_{33}\phi^{(1)} = 0 \tag{24d}$$

and the boundary conditions are

$$u^{(0)}_3 = 0, \quad \text{at } X_1 = 0, l, \tag{25a}$$

$$\widehat{T}^{(1)}_{11} = G^{(2)}_{1111}u^{(1)}_{1,1} + R^{(2)}_{111}\phi^{(1)}_{,1} = 0, \quad \text{at } X_1 = 0, l, \tag{25b}$$

$$\widehat{D}^{(0)}_{1} = -2h\bar{\varepsilon}_{33}\phi^{(0)}_{,1} = 0, \quad \text{at } X_1 = 0, l, \tag{25c}$$

$$\widehat{D}^{(1)}_{1} = R^{(2)}_{111}u^{(1)}_{1,1} - L^{(2)}_{11}\phi^{(1)}_{,1} = 0, \quad \text{at } X_1 = 0, l, \tag{25d}$$

where

$$
\begin{aligned}
&G^{(0)}_{1313} = 2hc_{44} + p, \quad G^{(0)}_{1331} = G^{(0)}_{3113} = G^{(0)}_{3131} = 2hc_{44},\\
&G^{(2)}_{1111} = \frac{1}{3}h^2(2h\bar{c}_{33} + p), \quad R^{(0)}_{313} = R^{(0)}_{331} = 2he_{15},\\
&R^{(2)}_{111} = \frac{2}{3}h^3\bar{e}_{33}, \quad L^{(2)}_{11} = \frac{2}{3}h^3\bar{\varepsilon}_{33}, \quad L^{(0)}_{33} = 2h\varepsilon_{11},\\
&\bar{c}_{33} = c_{33} - \frac{c_{13}^2}{c_{11}}, \quad \bar{e}_{33} = e_{33} - \frac{e_{31}c_{31}}{c_{11}}, \quad \bar{\varepsilon}_{33} = \varepsilon_{33} + \frac{e_{31}^2}{c_{11}}.
\end{aligned}
\tag{26}
$$

We note from (25b) that the bending moment $\widehat{T}^{(1)}_{11}$ is coupled to $W^{(1)}_1 = -\phi^{(1)}_{,1}$ through $\bar{e}_{33}$ as expected. Furthermore, from $\widehat{T}^{(0)}_{13} = G^{(0)}_{1313}u^{(0)}_{3,1} + G^{(0)}_{1331}u^{(1)}_1 + R^{(0)}_{313}\phi^{(1)}$ it can be seen that the transverse shear force $\widehat{T}^{(0)}_{13}$ is coupled to $W^{(0)}_3 = -\phi^{(1)}$ through e_{15} as expected. Eqs. (24c) and (25c) show that $\phi^{(0)}$ is a constant which can be taken to be zero. Let

$$
u^{(0)}_3 = A\sin\lambda X_1, \quad u^{(1)}_1 = B\cos\lambda X_1, \quad \phi^{(1)} = C\cos\lambda X_1,
\tag{27}
$$

where A, B, and C are undetermined constants and $\lambda = \pi/l$. The boundary conditions in (25a)–(25d) are satisfied automatically. Then the buckling load can be determined from the following equations obtained by substituting (27) into (24a), (24b) and (24d)

$$
\begin{aligned}
&\lambda^2 G^{(0)}_{1313}A + \lambda G^{(0)}_{1331}B + \lambda R^{(0)}_{313}C = 0,\\
&\lambda G^{(0)}_{3113}A + \left(\lambda^2 G^{(2)}_{1111} + G^{(0)}_{3131}\right)B + \left(\lambda^2 R^{(2)}_{111} + R^{(0)}_{331}\right)C = 0,\\
&\lambda R^{(0)}_{313}A + \left(\lambda^2 R^{(2)}_{111} + R^{(0)}_{331}\right)B - \left(\lambda^2 L^{(2)}_{11} + L^{(0)}_{33}\right)C = 0.
\end{aligned}
\tag{28}
$$

For nontrivial solutions of A, B, and$C$, the following condition must be satisfied

$$
\begin{vmatrix}
\lambda^2 G^{(0)}_{1313} & \lambda G^{(0)}_{1331} & \lambda R^{(0)}_{313}\\
\lambda G^{(0)}_{3113} & \lambda^2 G^{(2)}_{1111} + G^{(0)}_{3131} & \lambda^2 R^{(2)}_{111} + R^{(0)}_{331}\\
\lambda R^{(0)}_{313} & \lambda^2 R^{(2)}_{111} + R^{(0)}_{331} & -\left(\lambda^2 L^{(2)}_{11} + L^{(0)}_{33}\right)
\end{vmatrix} = 0,
\tag{29}
$$

which can be written as

$$
a^{(1)}\bar{p}^{(1)2} + b^{(1)}\bar{p}^{(1)} + c^{(1)} = 0,
\tag{30}
$$

where

$$
\begin{aligned}
&\bar{p}^{(1)} = \frac{p}{2h\bar{c}_{33}}, \quad \lambda_0 = \frac{\pi^2}{3}\left(\frac{h}{l}\right)^2, \quad a^{(1)} = \lambda_0,\\
&b^{(1)} = \lambda_0 + \frac{1}{\bar{c}_{33}}\left[(\lambda_0 + 1)c_{44} + \frac{\lambda_0 e_{15}^2 + \left(\lambda_0\bar{e}_{33} + e_{15}\right)^2}{\varepsilon_{11} + \lambda_0\bar{\varepsilon}_{33}}\right],\\
&c^{(1)} = \frac{1}{\bar{c}_{33}}\left[\lambda_0 c_{44} + \frac{\lambda_0 e_{15}^2 + \lambda_0^2\bar{c}_{33}^{-1}\bar{e}_{33}^2 c_{44}}{\varepsilon_{11} + \lambda_0\bar{\varepsilon}_{33}}\right].
\end{aligned}
\tag{31}
$$

Since $\lambda_0 \ll 1$ for a thin plate, it follows from (31) that $(b^{(1)})^2 \gg 4a^{(1)}c^{(1)}$ and $b^{(1)} \gg a^{(1)}$ for thin plates. Hence, an approximate solution of (30) can be found as

$$\bar{p}^{(1)} \cong -\frac{c^{(1)}}{b^{(1)}}\left(1+\frac{a^{(1)}c^{(1)}}{\left(b^{(1)}\right)^2}\right). \tag{32}$$

Our main interest is the effect of piezoelectric coupling on the buckling load. To see this more clearly, we let $c_{44} \to \infty$ in (32), which effectively eliminates plate shear deformations and the related piezoelectric coupling through e_{15}, and then expand (32) into a polynomial of λ_0 which is small. This leads to

$$\bar{p}^{(1)} \cong -\lambda_0\left(1+\lambda_0\bar{k}^2\right), \tag{33a}$$

$$\bar{k}^2 = \bar{e}_{33}^2/\left(\bar{c}_{33}\varepsilon_{11}\right), \tag{33b}$$

where $\bar{k}^2$ is an electromechanical coupling factor. We note that under our notation $-\lambda_0$ is the nondimensional buckling load from an elastic analysis without considering piezoelectric effect. The second term on the right hand side of (33a) represents the effect of piezoelectric coupling which is due to $\bar{e}_{33}$ alone now. We note that this additional term is proportional to $\bar{k}^2$ which ranges from 0.1 to 0.6 for most polarized ceramics. Since $\bar{k}^2$ is multiplied by λ_0 which is a small number, the piezoelectric modification on the buckling load is a small addition to the elastic buckling load. Hence an elastic analysis without considering piezoelectric effect gives a conservative estimate of the buckling load.

4.2. Case (b)

In this case we consider a ceramic bimorph as shown in Fig. 3(b). The end faces are unelectroded. Then the governing equations for the incremental fields are

$$G_{1313}^{(0)}u_{3,11}^{(0)} + G_{1331}^{(0)}u_{1,1}^{(1)} = 0, \tag{34a}$$

$$G_{1111}^{(2)}u_{1,11}^{(1)} + R_{111}^{(1)}\phi_{,11}^{(0)} - G_{3113}^{(0)}u_{3,1}^{(0)} - G_{3131}^{(0)}u_1^{(1)} = 0, \tag{34b}$$

$$R_{111}^{(1)}u_{1,11}^{(1)} - L_{11}^{(0)}\phi_{,11}^{(0)} = 0, \tag{34c}$$

$$L_{11}^{(2)}\phi_{,11}^{(1)} - L_{33}^{(0)}\phi^{(1)} = 0, \tag{34d}$$

with the following boundary conditions

$$u_3^{(0)} = 0, \quad \text{at } X_1 = 0, l, \tag{35a}$$

$$\widehat{T}_{11}^{(1)} = G_{1111}^{(2)}u_{1,1}^{(1)} + R_{111}^{(1)}\phi_{,1}^{(0)} = 0, \quad \text{at } X_1 = 0, l, \tag{35b}$$

$$\widehat{D}_1^{(0)} = R_{111}^{(1)}u_{1,1}^{(1)} - L_{11}^{(0)}\phi_{,1}^{(0)} = 0, \quad \text{at } X_1 = 0, l, \tag{35c}$$

$$\widehat{D}_1^{(1)} = -L_{11}^{(2)}\phi_{,1}^{(1)} = 0, \quad \text{at } X_1 = 0, l, \tag{35d}$$

where the plate material constants are as in (26) plus

$$R_{111}^{(1)} = h^2\bar{e}_{33}. \tag{36}$$

We note from (35b) that the bending moment $\widehat{T}_{11}^{(1)}$ is coupled to $W_1^{(0)} = -\phi_{,1}^{(0)}$ through $\bar{e}_{33}$ as expected. We also note that Eqs. (34d) and (35d) imply $\phi^{(1)} = 0$. Assume

$$u_3^{(0)} = A \sin \lambda X_1, \tag{37a}$$

$$u_1^{(1)} = B \cos \lambda X_1, \tag{37b}$$

$$\phi^{(0)} = D \cos \lambda X_1 \tag{37c}$$

which satisfy the boundary conditions in (35a)–(35d). Eliminating $\phi^{(0)}$ from (34b) by (34c), and substituting (37a) and (37b) into the resulting (34b) and (34a), we obtain

$$\begin{aligned} &\lambda^2 G_{1313}^{(0)} A + \lambda G_{1331}^{(0)} B = 0, \\ &\lambda G_{3113}^{(0)} A + \left\{ \lambda^2 \left[G_{1111}^{(2)} + (L_{11}^{(0)})^{-1} R_{111}^{(1)2} \right] + G_{3131}^{(0)} \right\} B = 0. \end{aligned} \tag{38}$$

For nontrivial solutions of A and B, the following must be true

$$\begin{vmatrix} \lambda G_{1313}^{(0)} & G_{1331}^{(0)} \\ \lambda G_{3113}^{(0)} & \lambda^2 \left[G_{1111}^{(2)} + (L_{11}^{(0)})^{-1} R_{111}^{(1)2} \right] + G_{3131}^{(0)} \end{vmatrix} = 0, \tag{39}$$

or

$$a^{(2)} \bar{p}^{(2)2} + b^{(2)} \bar{p}^{(2)} + c^{(2)} = 0, \tag{40}$$

where

$$\begin{aligned} &\bar{p}^{(2)} = \frac{p}{2h\bar{c}_{33}}, \quad a^{(2)} = \lambda_0, \\ &b^{(2)} = \lambda_0 + \frac{1}{\bar{c}_{33}} \left[(\lambda_0 + 1) c_{44} + \frac{3}{4} \lambda_0 \bar{e}_{33}^2 \bar{\varepsilon}_{33}^{-1} \right], \\ &c^{(2)} = \frac{1}{\bar{c}_{33}} \left[\lambda_0 c_{44} + \frac{3}{4} \lambda_0 c_{44} \bar{c}_{33}^{-1} \bar{e}_{33}^2 \bar{\varepsilon}_{33}^{-1} \right]. \end{aligned} \tag{41}$$

Then an approximate solution of (41) is

$$\bar{p}^{(2)} \cong -\frac{c^{(2)}}{b^{(2)}} \left(1 + \frac{a^{(2)} c^{(2)}}{(b^{(2)})^2} \right). \tag{42}$$

Letting $c_{44} \to \infty$ in (42) and then expanding the result into a polynomial of λ_0, we have

$$\bar{p}^{(2)} \cong -\lambda_0 \left(1 + \frac{3}{4} \bar{k}_{33}^2 \right), \tag{43a}$$

$$\bar{k}_{33}^2 = \bar{e}_{33}^2 / \left(\bar{c}_{33} \varepsilon_{33} \right). \tag{43b}$$

Comparing (43a) with (33a), we note the important difference that in (43a) the piezoelectric modification on the buckling load is not multiplied by the small number λ_0. This is because in (24a)–(24d) the plate piezoelectric coefficient $R_{111}^{(2)}$ is proportional to h^3 but in (34a)–(34d) $R_{111}^{(1)}$ is proportional to h^2 and the fact that it is the squares of the piezoelectric coefficients that appear in the buckling loads. Therefore the piezoelectric effect on the buckling load is much stronger in the present case than in the previous case.

4.3. Case (c)

The third example is a ceramic bimorph as shown in Fig. 3(c). For ceramics poled in the X_3 direction the material matrices are (Auld, 1973)

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{21} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{31} & c_{31} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{31} \\ 0 & 0 & e_{33} \\ 0 & e_{15} & 0 \\ e_{15} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{pmatrix}. \tag{44}$$

The plate is electroded at $X_1 = 0$ and $X_1 = l$, with shorted and grounded electrodes. The governing equations take the following form

$$G^{(0)}_{1313} u^{(0)}_{3,11} + G^{(0)}_{1331} u^{(1)}_{1,1} + R^{(1)}_{113} \phi^{(1)}_{,11} = 0, \tag{45a}$$

$$G^{(2)}_{1111} u^{(1)}_{1,11} - G^{(0)}_{3113} u^{(0)}_{3,1} - G^{(0)}_{3131} u^{(1)}_{1} + \left[R^{(1)}_{311} - R^{(1)}_{131}\right] \phi^{(1)}_{,1} = 0, \tag{45b}$$

$$L^{(0)}_{11} \phi^{(0)}_{,11} = 0, \tag{45c}$$

$$R^{(1)}_{113} u^{(0)}_{3,11} - \left[R^{(1)}_{311} - R^{(1)}_{131}\right] u^{(1)}_{1,1} - L^{(2)}_{11} \phi^{(1)}_{,11} + L^{(0)}_{33} \phi^{(1)} = 0. \tag{45d}$$

The boundary conditions are

$$u^{(0)}_3 = 0, \quad \text{at } X_1 = 0, l, \tag{46a}$$

$$\widehat{T}^{(1)}_{11} = G^{(2)}_{1111} u^{(1)}_{1,1} + h^2 \bar{e}_{31} \phi^{(1)} = 0, \quad \text{at } X_1 = 0, l, \tag{46b}$$

$$\phi^{(0)} = 0, \quad \text{at } X_1 = 0, l, \tag{46c}$$

$$\phi^{(1)} = 0, \quad \text{at } X_1 = 0, l, \tag{46d}$$

where

$$\begin{aligned} &G^{(2)}_{1111} = \frac{1}{3} h^2 \left(2h\bar{c}_{11} + p\right), \quad G^{(0)}_{1313} = 2hc_{44} + p, \\ &G^{(0)}_{1331} = G^{(0)}_{3113} = G^{(0)}_{3131} = 2hc_{44}, \quad R^{(1)}_{113} = R^{(1)}_{131} = h^2 e_{15}, \\ &R^{(1)}_{311} = h^2 \bar{e}_{31}, \quad L^{(0)}_{11} = 2h\varepsilon_{11}, \quad L^{(2)}_{11} = \frac{2}{3} h^3 \varepsilon_{11}, \quad L^{(0)}_{33} = 2h\bar{\varepsilon}_{33}, \\ &\bar{c}_{11} = c_{11} - c^2_{13}/c_{33}, \quad \bar{e}_{31} = e_{31} - e_{33}c_{31}/c_{33}, \quad \bar{\varepsilon}_{33} = \varepsilon_{33} + e^2_{33}/c_{33}. \end{aligned} \tag{47}$$

We note from (46b) that the bending moment $\widehat{T}^{(1)}_{11}$ is coupled to $W^{(0)}_3 = -\phi^{(1)}$ through $\bar{e}_{31}$ as expected. In this case $\phi^{(0)}$ is zero. We let

$$u^{(0)}_3 = A \sin \lambda X_1, \quad u^{(1)}_1 = B \cos \lambda X_1, \quad \phi^{(1)} = C \sin \lambda X_1, \tag{48}$$

which satisfy the boundary conditions (46a)–(46d). Substituting (48) into (45a),(45b) and (45d)

$$
\begin{aligned}
&\lambda^2 G_{1313}^{(0)} A + \lambda G_{1331}^{(0)} B + \lambda^2 R_{113}^{(1)} C = 0, \\
&\lambda G_{3113}^{(0)} A + \left(\lambda^2 G_{1111}^{(2)} + G_{3131}^{(0)}\right) B + \lambda\left(R_{131}^{(1)} - R_{311}^{(1)}\right) C = 0, \\
&\lambda^2 R_{113}^{(1)} A + \lambda\left(R_{131}^{(1)} - R_{311}^{(1)}\right) B - \left(\lambda^2 L_{11}^{(2)} + L_{33}^{(0)}\right) C = 0.
\end{aligned}
\tag{49}
$$

For nontrivial solutions of A, B, and C, the following condition should be satisfied

$$
\begin{vmatrix}
\lambda^2 G_{1313}^{(0)} & \lambda G_{1331}^{(0)} & \lambda^2 R_{113}^{(1)} \\
\lambda G_{3113}^{(0)} & \lambda^2 G_{1111}^{(2)} + G_{3131}^{(0)} & -\lambda\left(R_{311}^{(1)} - R_{131}^{(1)}\right) \\
\lambda^2 R_{113}^{(1)} & -\lambda\left(R_{311}^{(1)} - R_{131}^{(1)}\right) & -\left(\lambda^2 L_{11}^{(2)} + L_{33}^{(0)}\right)
\end{vmatrix} = 0,
\tag{50}
$$

or

$$
a^{(3)} \bar{p}^{(3)2} + b^{(3)} \bar{p}^{(3)} + c^{(3)} = 0,
\tag{51}
$$

where

$$
\begin{aligned}
&\bar{p}^{(3)} = \frac{p}{2h\bar{c}_{11}}, \quad a^{(3)} = \lambda_0, \\
&b^{(3)} = \lambda_0 + \frac{1}{\bar{c}_{11}} \left[(\lambda_0 + 1) c_{44} + \frac{3\lambda_0 \left[\lambda_0 e_{15}^2 + \left(e_{15} - \bar{e}_{31} \right)^2 \right]}{4\left(\lambda_0 \varepsilon_{11} + \bar{\varepsilon}_{33} \right)} \right], \\
&c^{(3)} = \frac{1}{\bar{c}_{11}} \left[\lambda_0 c_{44} + \frac{3\lambda_0 \left[\lambda_0 e_{15}^2 + \bar{c}_{11}^{-1} c_{44} \bar{e}_{31}^2 \right]}{4\left(\lambda_0 \varepsilon_{11} + \bar{\varepsilon}_{33} \right)} \right].
\end{aligned}
\tag{52}
$$

An approximate solution of (51) is found to be

$$
\bar{p}^{(3)} \cong -\frac{c^{(3)}}{b^{(3)}} \left(1 + \frac{a^{(3)} c^{(3)}}{\left(b^{(3)}\right)^2} \right).
\tag{53}
$$

Letting $c_{44} \to \infty$ in (53) and expanding it into a polynomial of λ_0, we have

$$
\bar{p}^{(3)} \cong -\lambda_0 \left(1 + \frac{3}{4} \bar{k}_{31}^2 \right), \quad \bar{k}_{31}^2 = \bar{e}_{31}^2 / \left(\bar{c}_{11} \bar{\varepsilon}_{33} \right)
\tag{54}
$$

which shows the same behavior as (43a) and (43b).

5. Conclusions

The coupled extension and flexure deformations with shear of an electroelastic plate under biasing fields can be described by a two-dimensional model. The application of this model in analyzing the buckling loads of three plates of different configurations shows that the electromechanical coupling strengthens the plates against buckling and that the strengthening effect is significant for materials of strong piezoelectric coupling. This implies that an elastic analysis without considering piezoelectric coupling yields a conservative estimate of the critical buckling loads.

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